and $\sum_{n=0}^{\infty} f(m,n) = AB$ (by thm 8.44) (4)

Multiplication of Series

Def: 8.45

Cauchy product of Two Series

Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{if } n = 0, 1, 2 .$$

Then Series $\sum_{n=1}^{\infty} c_n$ is called the cauchy product of $\sum a_n$ and $\sum b_n$

Theorem: 8.46 Merten's theorem on product of two Series:

Assume that $\sum_{n=0}^{\infty} a_n$ converges absolutely and has sum A and Suppose $\sum_{n=0}^{\infty} b_n$ converges with sum B. then the cauchy product of these two Series converges and has Sum AB.

Proof: Given (i) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(ii) $\sum_{n=0}^{\infty} b_n$ converges

(iii) $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$

TPT : The cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges and has sum AB.

The cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ if $n = 1, 2, 3 \ldots$

We define the partial sums.

$$A_n = \sum_{k=0}^{n} a_k$$

$$B_n = \sum_{k=0}^{n} b_k \quad \text{and} \quad B_{p-k} = \sum_{m=0}^{p-k} b_m$$

$$C_p = \sum_{n=0}^{p} c_n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Let us define

$$d_n = B - B_n \longrightarrow ②$$

and

$$e_n = \sum_{k=0}^{n} a_k d_{n-k} \longrightarrow ③$$

Now we consider the partial sum of the Cauchy product

$$c_p = \sum_{n=0}^{p} c_n = \sum_{n=0}^{p} \sum_{k=0}^{p} a_k b_{n-k}$$

$$c_p = \sum_{n=0}^{p} \sum_{k=0}^{p} f_n(k) \longrightarrow ④$$

where,

$$f_n(k) = \begin{cases} a_k b_{n-k} & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$$

④ $\Rightarrow$

$$c_p = \sum_{n=0}^{p} \sum_{k=0}^{p} f_n(k)$$

$$= \sum_{k=0}^{p} \sum_{n=0}^{p} f_n(k)$$

$$= \sum_{k=0}^{p} \left\{ \sum_{n=0}^{k-1} f_n(k) + \sum_{n=k}^{p} f_n(k) \right\}$$

$$= \sum_{k=0}^{p} \left\{ 0 + \sum_{n=k}^{p} a_k b_{n-k} \right\}$$

$n-k=m$

$$= \sum_{k=0}^{p} \sum_{n=k}^{p} a_k b_{n-k}$$

Put $n=k \Rightarrow m=0$

Put $n=p \Rightarrow m=p-k$

$$= \sum_{k=0}^{p} \sum_{m=0}^{p-k} a_k b_m$$

$$= \sum_{k=0}^{p} a_k \sum_{m=0}^{p-k} b_m$$

$$= \sum_{k=0}^{p} a_k B_{p-k} \qquad (\text{by } ②)$$

$$= \sum_{k=0}^{p} a_k (B - d_{p-k})$$

$$= \sum_{k=0}^{p} a_k B - \sum_{k=0}^{p} a_k d_{p-k}$$

$$\left[\because e_n = \sum_{k=0}^{n} a_k d_{n-k} \right]$$

$$= B \sum_{k=0}^{p} a_k - e_p \qquad (\text{by } ③)$$

$$c_p = BA_p - e_p$$

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$$\Rightarrow \lim_{p\to\infty} c_p = \lim_{p\to\infty} BA_p - \lim_{p\to\infty} e_p$$

$$= B\lim_{p\to\infty} A_p - \lim_{p\to\infty} e_p$$

$$\Rightarrow \lim_{p\to\infty} c_p = BA - \lim_{p\to\infty} e_p \longrightarrow (5)$$

Since given that $\sum_{n=1}^{\infty} a_n = A$, the sequence is partition sums $\{A_n\}$ converges to A.

(5) $\Rightarrow$ the sequence is partition sums $\{\cdot\{c_p\} \longrightarrow AB$ is the sequence $\{e_p\} \longrightarrow 0$

i.e) The Cauchy product of the two given series converges and has sum AB if we p.T

$$e_p \longrightarrow 0 \quad \text{as } p \longrightarrow \infty$$

i.e) we have to p.T

for every given $\varepsilon > 0$ $\exists$ l $\ni$ $|e_p| < \varepsilon$

Since $\sum b_n = B$ and since $d_n = B - B_n$

$\{d_n\}$ converges to 0

Since $\{d_n\}$ is convergent it is bounded

we can choose M such that

$$|d_n| \leq M \quad \text{for all } n$$

Since $\sum a_n$ converges absolutely,

$$\text{Let } \sum_{n=0}^{\infty} |a_n| = K$$

Since $\{d_n\} \longrightarrow 0$ we have

for any given $\varepsilon > 0$, we can find N such that

$$|d_n| < \frac{\varepsilon}{2K} \quad \text{whenever } n > N \longrightarrow (7)$$

Also, using Cauchy condition for the series $\sum |a_n|$ we have,

$$\sum_{n=n+1}^{\infty} |a_n| < \frac{\varepsilon}{2M} \longrightarrow (8)$$

Then, for $p > 2n$ we have.

$$|e_p| = \left|\sum_{k=0}^{p} a_k d_{p-k}\right|$$

$$= \left|\sum_{k=0}^{N} a_k d_{p-k} + \sum_{k=N+1}^{p} a_k d_{p-k}\right|$$

$$\leq \sum_{k=0}^{N} |a_k d_{p-k}| + \sum_{k=N+1}^{p} |a_k d_{p-k}|$$

$$= \sum_{k=0}^{N} |a_k|\,|d_{p-k}| + \sum_{k=N+1}^{p} |a_k|\,|d_{p-k}|$$

$$< \sum_{k=0}^{N} |a_k| \frac{\epsilon}{2K} + \sum_{k=N+1}^{p} |a_k| \cdot M$$

$$= \frac{\epsilon}{2K} \sum_{k=0}^{N} |a_k| + M \sum_{k=N+1}^{p} |a_k| \qquad [\text{by } (5) \,\&\, (6)]$$

$$\leq \frac{\epsilon}{2K} \sum_{k=0}^{\infty} |a_k| + M \sum_{k=N+1}^{\infty} |a_k|$$

$$< \frac{\epsilon}{2K} \cdot K + M \cdot \frac{\epsilon}{2M} \qquad [\text{by } (8)]$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow |e_p| < \epsilon$$

This proves that $e_p \to 0$ as $p \to \infty$

Hence $c_p \to AB$ as $p \to \infty$.

Dirichlet product

Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

Define c_n by $c_n = \sum_{d/n} a_d\, b_{n/d}$, $n = 1, 2 \ldots$

where $\sum_{d/n}$ means the sum extended over all positive divisors of n.

For eg: $c_1 = a_1 b_1$

$c_2 = a_1 b_2 + a_2 b_1$

$c_3 = a_1 b_3 + a_3 b_1$

$c_4 = a_1 b_4 + a_2 b_2 + a_4 b_1$

$c_5 = a_1 b_5 + a_5 b_1$

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Then, for $p \geq 2N$ we have

$$|e_p| = \left|\sum_{k=0}^{p} a_k d_{p-k}\right|$$

$$= \left|\sum_{k=0}^{N} a_k d_{p-k} + \sum_{k=N+1}^{p} a_k d_{p-k}\right|$$

$$\leq \sum_{k=0}^{N} |a_k d_{p-k}| + \sum_{k=N+1}^{p} |a_k d_{p-k}|$$

$$= \sum_{k=0}^{N} |a_k|\,|d_{p-k}| + \sum_{k=N+1}^{p} |a_k|\,|d_{p-k}|$$

$$< \sum_{k=0}^{N} |a_k| \cdot \frac{\epsilon}{2K} + \sum_{k=N+1}^{p} |a_k| \cdot M$$

[by (1) & (2)]

$$= \frac{\epsilon}{2K} \sum_{k=0}^{N} |a_k| + M \sum_{k=N+1}^{p} |a_k|$$

$$\leq \frac{\epsilon}{2K} \sum_{k=0}^{\infty} |a_k| + M \sum_{k=N+1}^{\infty} |a_k|$$

$$< \frac{\epsilon}{2K} \cdot K + M \cdot \frac{\epsilon}{2M} \quad [\text{by } (3)]$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\Rightarrow |e_p| < \epsilon$

This proves that $e_p \to 0$ as $p \to \infty$

Hence $c_p \to AB$ as $p \to \infty$

Dirichlet product

Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

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For eg:

$c_1 = a_1 b_1$

$c_2 = a_1 b_2 + a_2 b_1$

$c_3 = a_1 b_3 + a_3 b_1$

$c_4 = a_1 b_4 + a_2 b_2 + a_4 b_1$

$c_5 = a_1 b_5 + a_5 b_1$

$$c_6 = a_1 b_6 + a_2 b_3 + a_3 b_2 + a_6 b_1$$

and $$c_7 = a_1 b_7 + a_7 b_1$$

Dirichelt Series

The Series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is called a dirichlet Series

Note $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ are two absolutely convergent dirichlet series having sums $A(s)$ and $B(s)$ respectively

Then $\sum_{n=1}^{\infty} \frac{c_n}{n^s} = A(s)\, B(s)$ where $c_n = \sum_{d/n} a_d \, b_{n/d}$

8.25 Cesaro Summability

Defn: 8.47

Let s_n denote the n^{th} partial sum of the Series $\sum a_n$ and let $\{\sigma_n\}$ be the sequence of arithmetic means defined by $\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$, if $n = 1, 2, \ldots$ The Series $\sum a_n$ is said to be cesaro summable (or) $(C,1)$ summable if $\{\sigma_n\}$ converges

If $\lim_{n \to \infty} \sigma_n = S$, then S is called the cesaro Sum (or) $(C,1)$ sum of $\sum a_n$, and we write

$$\sum a_n = S \ (C,1)$$

Example: 1

Let $a_n = z^{n-1}$, $|z| < 1$, $z \neq 1$

find the cesaro sum of $\sum a_n$

To find s_n:

$$s_n = \sum_{k=1}^{n} a_k$$

$$= a_1 + a_2 + a_3 + \cdots + a_n$$

$$= 1 + z + z^2 + \cdots + z^{n-1}$$

$$= \frac{1 - z^n}{1 - z}$$

$$s_n = \frac{1}{1-z} - \frac{z^n}{1-z}$$

and $\sigma_n = \dfrac{s_1 + s_2 + \cdots + s_n}{n}$

$$= \frac{\left(\frac{1}{1-z} - \frac{z}{1-z}\right) + \left(\frac{1}{1-z} - \frac{z^2}{1-z}\right) + \cdots + \left(\frac{1}{1-z} - \frac{z^n}{1-z}\right)}{n}$$

$$= \frac{1}{1-z} - \frac{z}{n(1-z)}\left(1 + z + z^2 + \cdots + z^{n-1}\right)$$

$$= \frac{1}{1-z} - \frac{z}{n(1-z)}\left(\frac{1-z^n}{1-z}\right)$$

$$= \frac{1}{1-z} - \frac{z(1-z^n)}{n(1-z)^2} \longrightarrow (1)$$

$\therefore \sigma_n = \dfrac{1}{1-z} - \dfrac{z}{(1-z)^2} \cdot \dfrac{(1-z^n)}{n}$

Since $|z| < 1$, $\dfrac{1-z^n}{n} \longrightarrow 0$ as $n \to \infty$

$\therefore$ From (1) $\sigma_n \longrightarrow \dfrac{1}{1-z}$ as $n \to \infty$

$\therefore$ The series $\sum_{n=1}^{\infty} z^{n-1}$ is Cesaro summable with Cesaro sum (ie) $(C,1)$ sum

ie) $\sum_{n=1}^{\infty} z^{n-1} = \dfrac{1}{1-z}$ $(C,1)$

In particular if $z = -1$ then

$\sum_{n=1}^{\infty} (-1)^{n-1}$ is Cesaro summable with Cesaro Sum $1/2$

ie; $\sum_{n=1}^{\infty} (-1)^{n-1} = \frac{1}{2}$ $(C,1)$

Example : 2

Let $a_n = (-1)^{n+1} \cdot n$ P.T $\sum (-1)^{n+1} \cdot n$ is not Cesaro summable (or) $(C,1)$ summable.

Solution:

$s_1 = a_1 = 1$

$s_2 = a_1 + a_2$

$= 1 - 2 = -1$

$s_3 = a_1 + a_2 + a_3$

$= 1 - 2 + 3 = 2$

$s_4 = a_1 + a_2 + a_3 + a_4$

$= s_3 + a_4 = 2 - 4 = -2$

$s_5 = s_4 + a_5 = -2 + 5 = 3$

$s_6 = s_5 + a_6 = 3 - 6 = -3$

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If n is even $s_1 + s_2 + \cdots + s_n = 0$

If n is odd $\Rightarrow s_1 + s_2 + \cdots + s_n = s_n$

$s_1 + s_2 + \cdots + s_n = s_n$

$$\Rightarrow \sigma_n = \frac{s_n}{n} \qquad \left[\because \sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}\right]$$

If $n = 2k-1$

$$\sigma_{2k+1} = \frac{s_{2k-1}}{2k-1} = \frac{k}{2k-1}$$

$$\Rightarrow \frac{k}{2k-1} = \frac{1}{2}\left[\frac{2k}{2k-1}\right]$$

$$= \frac{1}{2}\left[\frac{2k-1+1}{2k-1}\right]$$

$$= \frac{1}{2}\left[\frac{2k-1}{2k-1} + \frac{1}{2k-1}\right]$$

$$\frac{k}{2k-1} = \frac{1}{2}\left[1 + \frac{1}{2k-1}\right]$$

$$\therefore \sigma_{2k-1} = \frac{k}{2k-1} = \frac{1}{2}\left[1 + \frac{1}{2k-1}\right]$$

$$\lim_{k\to\infty} \sup(\sigma_{2k-1}) = \lim_{k\to\infty} \frac{1}{2}\left[1 + \frac{1}{2k-1}\right] = \frac{1}{2}(1+0)$$

$$\lim_{k\to\infty} \sup(\sigma_{2k+1}) = \frac{1}{2}$$

$$\lim_{k\to 0} \inf(\sigma_{2k-1}) = \lim_{k\to 0} \frac{1}{2}\left[1 + \frac{1}{2k-1}\right]$$

$$= \frac{1}{2}[1-1]$$

$$\lim_{k\to 0} \inf(\sigma_{2k-1}) = 0$$

$$\lim \sup(\sigma_{2k-1}) \neq \lim \inf(\sigma_{2k-1})$$

$\therefore \sigma_{2k-1}$ does not converge

Hence $\sum (-1)^{n+1} \cdot n$ is not $(C,1)$ Summable.

Theorem: 8·48

If a Series is convergent with sum S, then it is also $(C,1)$ summable with Cesaro Sum S.

proof: Given: $\sum_{n=1}^{\infty} a_n$ is convergent with sum S.

To prove: $\sum_{n=1}^{\infty} a_n$ is $(C,1)$ summable with sum S

Let s_n denote the n^{th} partial sum of the series then $s_n \to S$ as $n \to \infty$

Define σ_n by

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n} \quad \text{if } n = 1, 2, \ldots$$

Introduce $t_n = s_n - S$

$T_n = \sigma_n - S$

ie) $$T_n = \frac{s_1 + s_2 + \cdots + s_n}{n} - S$$

$$= \frac{t_1 + S + t_2 + S + \cdots + t_n + S}{n} - S$$

$$= \frac{t_1 + t_2 + \cdots + t_n + nS}{n} - S$$

$$= \frac{t_1 + t_2 + \cdots + t_n}{n} + \frac{nS}{n} - S$$

$$= \frac{t_1 + t_2 + \cdots + t_n}{n} \longrightarrow (1)$$

TPT : The Series is $(C,1)$ summable with $(C,1)$ sum S.

We have to P.T

$$\lim_{n \to \infty} \sigma_n = S$$

ie) T.P.T : $\sigma_n \to S$ as $n \to \infty$

For this we have T.P.T $T_n = 0$ as $n \to \infty$

T.P.T $T_n \to 0$ as $n \to \infty$

Since $t_n = s_n - S$

$t_n \to 0$ as $n \to \infty$

$\therefore \{t_n\}_{n=1}^{\infty}$ converges

$\Rightarrow \{t_n\}_{n=1}^{\infty}$ is bounded

We can find $A > 0$ such that

$|t_n| \leq A \quad \forall n \longrightarrow (2)$

Since $\{t_n\}_{n=1}^{\infty}$ is a convergent sequence, converges to zero for any given $\epsilon>0$, we can find N such that

$$|t_n| < \epsilon/2 \quad \forall\, n \geq N$$

For $n>N$

$$|T_n| = \frac{|t_1+t_2+\cdots+t_n|}{n}$$

$$= \frac{|t_1+t_2+\cdots+t_N+t_{N+1}+\cdots+t_n|}{n}$$

$$\leq \frac{|t_1|+|t_2|+\cdots+|t_N|}{n} + \frac{|t_{N+1}|+\cdots+|t_n|}{n}$$

$$< \frac{A+A+\cdots+A \text{ (N times)}}{n} + \frac{\frac{\epsilon}{2}+\frac{\epsilon}{2}+\cdots+\frac{\epsilon}{2}\,(n-N) \text{ times}}{n}$$

$$= \frac{NA}{n} + \frac{(n-N)}{n}\frac{\epsilon}{2}$$

$$= \frac{NA}{n} + \left(1-\frac{N}{n}\right)\frac{\epsilon}{2}$$

$$< \frac{NA}{n} + \frac{\epsilon}{2}$$

$$\Rightarrow |T_n| < \frac{NA}{n} + \epsilon$$

Hence $\lim_{n\to\infty} \sup |T_n| < \epsilon$

Since $\epsilon>0$ is arbitrary

$$\lim_{n\to\infty} |T_n| = 0$$

8.26 Infinite product

Def:

Given a sequence $\{u_n\}$ of real or complex numbers

Let, $P_1 = u_1$

$P_2 = u_1 u_2$

$P_3 = u_1 u_2 u_3$

$$P_n = u_1 u_2 \cdots u_n = \prod_{k=1}^{n} u_k \longrightarrow (1)$$

The ordered pair of sequence $(\{u_n\}, \{P_n\})$ is called an infinite product or simply a product the number P_n is called the n^{th} partial product and u_n is